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Freya: An educational MATLAB GUI-based tool for generalized Fourier series

Freya: Uma ferramenta educacional em Ambiente GUIDE do MATLAB para séries de Fourier generalizadas

Freya: Una herramienta educativa en el Entorno GUIDE de MATLAB para series de Fourier generalizadas

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Abstract
The Fourier analysis is a very powerful mathematical tool to decompose functions into their frequency components. Due to this, it has applications in a wide variety of fields inside the realm of science and engineering. As usual, this theory starts with a discussion about the trigonometric Fourier series, the expansion of a function in terms of sines and cosines, and then is generalized in the sense that other functions rather than the trigonometric ones can be used as an orthogonal basis, as the eigenfunctions of some specific Sturm-Liouville problems, such as Bessel functions and Legendre polynomials. In this direction, we present the so- called Freya, an educational graphical user interface (GUI) for the generalized Fourier series developed using the interactive MATLAB (MATrix Laboratory) App Designer environment. We aim to provide a user-friendly tool as a learning aid system for students to gain a comprehensive understanding of the subject as well as for teaching.
Keywords: Generalized Fourier series; Sturm-Liouville problems; Besselfunctions; Legendre polynomials; MATLAB.

Resumo
A análise de Fourier é uma ferramenta matemática muito poderosa para decompor funções em seus componentes de frequência. Devido a isso, tem aplicações em uma variedade ampla de áreas dentro do domínio da ciência e da engenharia. Como de costume, essa teoria começa com uma discussão sobre a série trigonométrica de Fourier, a expansão de uma função em termos de senos e cossenos, e depois é generalizada no sentido de que outras funções, além das trigonométricas, podem ser usadas como base ortogonal, como autofunções de alguns problemas específicos de Sturm-Liouville, como funções de Bessel e polinômios de Legendre. Nesse sentido, apresenta-se a chamada Freya, uma interface gráfica do usuário (GUI) educacional para a série de Fourier generalizada desenvolvida usando o ambiente interativo MATLAB (MATrix LABoratory) App Designer. Nosso objetivo é fornecer uma ferramenta amigável como um sistema de auxílio ao aprendizado para que os alunos obtenham uma compreensão abrangente do assunto, bem como para o ensino.
Palavras-chave: Série de Fourier generalizada; Problemas de Sturm-Liouville; Funções de Bessel; Polinômios de Legendre; MATLAB.

Resumen
El análisis de Fourier es una herramienta matemática muy poderosa para descomponer funciones en sus componentes de frecuencia. Por esta razón, la herramienta se puede aplicar en una amplia variedad de áreas dentro del dominio de la ciencia y la ingeniería. Como de costumbre, esta teoría comienza con una discusión de la serie trigonométrica de Fourier, la expansión de una función en términos de senos y cosenos, y luego se generaliza en el sentido de que funciones

distintas a las trigonométricas se pueden usar como base ortogonal, como por ejemplo: funciones propias de algunos problemas específicos de Sturm-Liouville, como las funciones de Bessel y los polinomios de Legendre. En este sentido, presentamos "Freya", una interfaz gráfica de usuario (GUI) educativa para la serie de Fourier generalizada desarrollada utilizando el entorno interactivo MATLAB (MATrix LABoratory) App Designer. Nuestro objetivo es proporcionar una herramienta fácil de usar como un sistema de ayuda al aprendizaje para que los estudiantes obtengan una comprensión integral de la materia, así como para la enseñanza.
Palabras clave: Series de Fourier generalizadas; Problemas de Sturm-Liouville; Funciones de Bessel; Polinomios de Legendre; MATLAB.

1. Introduction

1.1 Historical Prologue

Jean Baptiste Joseph Fourier was born in poor circumstances in France on March 21, 1768, in the town of Auxerre (Bracewell, 1985). His father, from whom he inherited the name, was a tailor who married twice. In the second marriage, he had twelve children and Fourier was ninth among them. Unfortunately, Fourier becomes an orphan before completing ten years old, since his mother died in 1777 and his father in the following year (Prestini, 2016). At the age of 12, he entered the École Royale Militaire and very soon his talent was evidenced in several areas of knowledge, in particular for Mathematics. At the age of seventeen, his ambition was the military career, but his application to enter the artillery or the engineers were rejected by the minister of war, probably because he was not a noble (Prestini, 2016). In his twenties, Fourier was involved for the first time with politics in a troubled period of French history.

The end of the eighteenth century is a period characterized by economic, social, and political instabilities due to the fall of the ancient regime, as a result of the French Revolution. Fourier went through the Revolution as a convinced Jacobin, being a member of the Comité de Surveillance (1793) and the president of the Revolutionary Committee (1794), both in Auxerre. He believed in the ideals of the Revolution, as he demonstrated in some letters, but always retained an independent judgment against the excesses of the Revolution (Prestini, 2016). In particular, Fourier was assigned to a commission in Orleans and became involved in a local dispute against the abuse of power of the Convention in that town. As a result, he was denounced to the Convention and declared incapable of receiving such commissions in the future (Prestini, 2016). Feeling injustice by this declaration, he goes in person to Robespierre in Paris to pledge for his cause, being arrested on his return to Auxerre on July 4, 1794. Because of the fall of Robespierre on July 27, Fourier regained his freedom, owing to this episode both life and liberty. After this, Fourier was arrested a second time under the charges of inspiring terrorism during the Terror (Maurey, 2019).

Soon after being released from his first imprisonment, Fourier is chosen by the St. Florentin district to be one of the 1500 students in the newly École Normale in Paris, created by a decree from October 30, 1794, to conceive a new educational system in agreement with the ideals of the Revolution. As a student at École Normale, Fourier had contact with some of the most renowned mathematicians in the country, namely Joseph Louis Lagrange (1736-1813), Pierre Simon Laplace (1749-1827), and Gaspard Monge (1746-1816), who in the future were going to play a role in Fourier's life (Prestini, 2016). Under the influence of Monge, Fourier is appointed to an assistant teaching post in the École Centrale, which by a decree from September 1, 1795, is renamed to École Polytechnique. There, he became involved in several activities beyond teaching, publishing his first paper in 1798.

On March 1798, the minister of the interior invited Fourier to be part of the Commission of Arts and Sciences of the expedition of general Napoleon Bonaparte in Egypt (Scharlau & Opolka, 1985). He spent 3 years there and was assigned to deal with several administrative and judicial issues, earning the confidence of Napoleon. In particular, he helped to organize an expedition to Upper Egypt whose discoveries served as a basis to conceive the manuscript Description of Egypt, in which Fourier provided a historical introduction (Prestini, 2016). In November 1801, Fourier returned to Paris and resumed his teaching position at École Polytechnique, but on February 1802 Napoleon appointed him to be the prefect of the department of Isère (González-Velasco, 1996).

It is outstanding that while mayor he found time to work on the theory of the conduction of heat, writing a memoir in 1807 that he presented to the Institut de France (Grattan-Guinness, 2005). Four mathematicians were appointed to examine Fourier's paper, three of them - Lagrange, Laplace, and Monge - had already a high opinion of Fourier as mentioned earlier. Due to objections of the referees, who criticized the lack of rigor regarding the derivation of the heat equation and the use of trigonometric series to represent arbitrary functions, the paper was not published (Boyce et al., 2001). Fourier extended and improved it for the occasion of the grand prize in mathematics, promoted by the Institut in 1812, and submitted a new memoir at the end of 1811, as a candidate for the prize (Carslaw, 1950). Despite having won, Fourier's memorial was not published under similar criticism from the referees. As he understood that the Institut de France was in no hurry to publish his work, he wrote another version of it as a book entitled Analytical Theory of Heat, published only on 1822 (Herivel & Williams, 1975).

The problem of the representation of functions by trigonometric series remained open from the middle of the 18th century until 1829, when the German mathematician Johann Peter Gustav Lejeune Dirichlet (1805 - 1859) introduced the sufficient conditions for the convergence of the Fourier series (Prestini, 2016), since Fourier's argument was still imprecise in those days. Fourier died on May 4, 1830, leaving behind a theory that has very important applications on modern subjects such as electrical engineering, e.g. signal processing and circuit network analysis (Hmurcik et al., 2000; Reed et al., 1990), and physics, e.g. quantum mechanics and electromagnetism (Arrayás & Trueba, 2018; Do, 2018; Horwitz, 2020;), to name just a few. The success of the ideas introduced by Fourier had a profound impact on mathematics (Prestini & Prestini, 2004), as they required a redefinition of the concept of function, the introduction of a precise notion of convergence and a reexamination of the concept of integral (Debnath, 2012).

1.2 Goals and Paper's Structure

The main purpose of this paper is to present a new MATLAB GUI-based tool for generalized Fourier series and show its capabilities and functionalities that can be explored for educational purposes. Also, it is intended to describe the underlying mathematics used by the authors of this text to implement the tool using a programming language. The motivation for the development of Freya is to disseminate the Fourier Analysis in a more didactic way due to its practical importance, as mentioned in Subsection 1.1.

The remainder of this paper is structured as follows. Section 2 presents a review on generalized Fourier series that arises in the context of Sturm-Liouville problems (SLP) and three specific cases are studied: trigonometric Fourier series, Fourier-Legendre series, and Fourier-Bessel series. In each one, the orthogonality of the basic functions is shown and some formulas are obtained. Section 3 details Freya, an educational tool to assist undergraduate students attending classes that rely on this theory, especially in computer and electrical engineering courses. The paper is concluded in Section 4 with some concluding remarks regarding the obtained results and the possible directions for future work.

2. Methodology

This section describes the generalized Fourier series to address the problem of expanding a function in an infinite set of orthogonal basis functions. To find such a basis, one possibility is to use the eigenfunctions of specific Sturm-Liouville problems. This idea can be applied to the trigonometric Fourier series, Fourier- Legendre series, and Fourier-Bessel series, as presented in subsections 2.2, 2.3, and 2.4, respectively. Some alternative representations of the trigonometric Fourier series are also discussed, namely, the exponential and polar Fourier series since they will play an important role in Section 3.

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2.1 Generalized Fourier Series

Let $f: [a, b] \to \mathrm{R}$, with $a < b$, and $B = \{\varphi_n(\mathrm{x})\}_{\mathrm{n}=0}^{\infty}$ be an infinite orthogonal set of functions on the same interval with respect to the weight function $w(x) > 0$ on $[a, b]$. Thus, it follows by definition that

()

$$\langle \varphi_n, \varphi_m \rangle = \int_a^b w(x)\varphi_n(x)\varphi_m(x)dx = \begin{cases} 0, & n \neq m \\ \|\varphi_n(x)\|^2, & n = m \end{cases}, \qquad (1)$$

where $\langle \cdot,\cdot \rangle$ denotes a weighted inner product, $\|\cdot\|$ the corresponding induced norm, and $m, n \in \mathbb{N}$. Note that φ_n cannot be identically zero since B is linearly independent. Suppose that f can be expanded as a linear combination of the orthogonal functions φ_n, i.e.,

$$f(x) = \sum_{n=0}^{\infty} c_n\, \varphi_n(x), \qquad (2)$$

where c_n are the coefficients of φ_n. Now, the problem consists on finding these coefficients. We can use the orthogonality of the functions φ_n to search for an explicit formula for c_n. In fact, by taking the inner product $\langle f, \varphi_m \rangle$ and using the orthogonal relation in Eq. (1), we obtain

$$\begin{aligned} \langle f, \varphi_m \rangle &= \int_a^b w(x)f(x)\varphi_m(x)dx \\ &\overset{(2)}{=} \sum_{n=0}^{\infty} c_n \underbrace{\int_a^b w(x)\varphi_n(x)\varphi_m(x)dx}_{(1)} \\ &= c_n\|\varphi_n(x)\|^2 \Longrightarrow c_n = \frac{\langle f, \varphi_n \rangle}{\|\varphi_n(x)\|^2}. \qquad (3) \end{aligned}$$

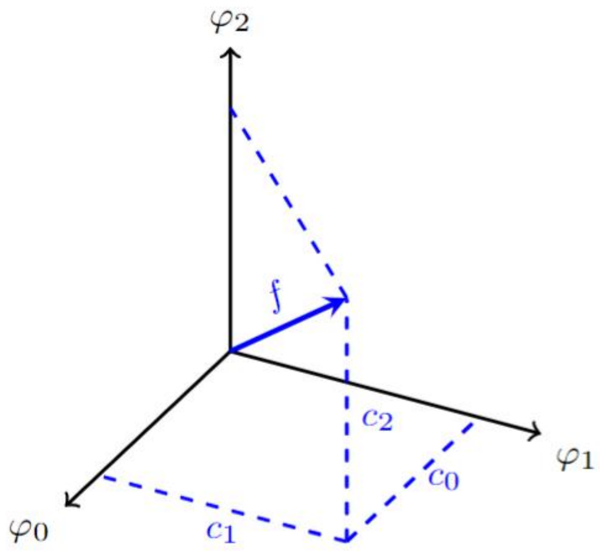
In the context of inner product spaces, we conclude that the coefficients c_n as defined in Eq. (3) are the orthogonal projections of f onto the basis functions φ_n, as shown in Figure 1 for the first three functions φ_0, φ_1 and φ_2.

Substituting the coefficients c_n formula in Eq. (3) into (2) yields

$$f(x) = \sum_{n=0}^{\infty} \frac{\langle f, \varphi_n \rangle}{\|\varphi_n(x)\|^2} \varphi_n(x), \qquad (4)$$

which is a closed expression to expand f in an orthogonal series. The series in Eq. (4) is called generalized Fourier series for the function f on the interval $[a, b]$, with respect to the orthogonal basis B (Brown & Churchill, 2008). The coefficients c_n are known as Fourier coefficients. Thus, there are several types of Fourier series expansions depending on the chosen basis. If B is defined in terms of a trigonometric system (set of orthogonal trigonometric functions), the series in Eq. (4) is called trigonometric Fourier series (see subsection 2.2), while if it is defined in terms of the Legendre polynomials, it is called Fourier-Legendre series (see subsection 2.3). Similarly, if B is defined in terms of the Bessel functions, then it will be called Fourier-Bessel series (see subsection 2.4). There are other families of orthogonal functions (e.g., Chebyshev and Hermite polynomials) but our goal is to treat only the aforementioned ones. It is worth to mention that in the entire discussion B is considered to be complete in the sense that f is not orthogonal to each basis function in B.

Figure 1 - Geometric interpretation of the coefficients c_n for the first three basis functions.



Source: Authors.

Before delving into each type of Fourier series considered in this paper, it is necessary to mention how to find an orthogonal basis for the generalized Fourier series in Eq. (4). To do this, one can resort to the Sturm-Liouville theory, which deals with second-order ODEs of the form

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0 \qquad (5)$$

on some interval $[a, b]$, satisfying the following homogeneous boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad \alpha_1^2 + \alpha_2^2 > 0, \qquad (6a)$$
$$\beta_1 y(b) + \beta_2 y'(b) = 0, \quad \beta_1^2 + \beta_2^2 > 0, \qquad (6b)$$

where α_1, α_2, β_1, and β_2 are given real constants, λ is a parameter, and p, q and r are given functions. The Eq. (5) together with the conditions in Eqs. (6a) and (6b) compose a boundary value problem (BVP) called Sturm-Liouville problem (SLP). A SLP is said to be *regular* if p, p', q, and r are continuous functions on an interval $[a, b]$, and $p(x), r(x) > 0$ for every x in the interval. Otherwise, it is called *singular*. The non-trivial solutions of a SLP are called *eigenfunctions* and the values of λ for which an eigenfunction exists are called *eigenvalues*. For each eigenvalue there is only one eigenfunction (except for nonzero constant multiples). An interesting result of the Sturm-Liouville theory is that the set of eigenfunctions of a regular SLP corresponding to the set of eigenvalues is orthogonal with respect to the weight function $r(x)$ on the interval $[a, b]$, *i.e.*,

$$\int_a^b r(x) y_m(x) y_n(x) dx = 0, \qquad m \neq n, \qquad (7)$$

where y_m and y_n are eigenfunctions. To see this, let λ_m and λ_n be the corresponding distinct eigenvalues. By assumption, y_m and y_n satisfy the Sturm-Liouville equations

$$[p(x)y_m']' + [q(x) + \lambda_m r(x)]y_m = 0. \qquad (8)$$
$$[p(x)y_n']' + [q(x) + \lambda_n r(x)]y_n = 0. \qquad (9)$$

Multiplying Eq. (8) by y_n and Eq. (9) by $-y_m$ and adding the resulting equations gives

$$(\lambda_m - \lambda_n) r(x) y_m y_n = [(p(x)y_n')y_m - (p(x)y_m')y_n]'. \qquad (10)$$

Integrating both sides of Eq. (10) with respect to x from a to b yields

$$(\lambda_m - \lambda_n)\int_a^b r(x)y_m(x)y_n(x)dx = p(b)[y_n'(b)y_m(b) - y_m'(b)y_n(b)]$$
$$-p(a)[y_n'(a)y_m(a) - y_m'(a)y_n(a)]. \qquad (11)$$

The eigenfunctions y_m and y_n satisfy the boundary conditions in Eqs. (6a) and (6b). In particular, for $x = a$ one can obtain the following system of equations

$$\begin{cases} \alpha_1 y_m(a) + \alpha_2 y_m'(a) = 0, \\ \alpha_1 y_n(a) + \alpha_2 y_n'(a) = 0. \end{cases}$$

Since by hypothesis α_1 and α_2 cannot be zero simultaneously, this homogeneous system must have infinite solutions. Thus, the determinant of the coefficients must be zero, *i.e.*,

$$y_n'(a)y_m(a) - y_m'(a)y_n(a) = 0. \qquad (12)$$

Similarly, for $x = b$ it follows that

$$y_n'(b)y_m(b) - y_m'(b)y_n(b) = 0. \qquad (13)$$

Substituting Eqs. (12) and (13) in the right-hand side of Eq. (11) yields the orthogonality relation in Eq. (7) since $\lambda_m \neq \lambda_n$. As one can notice in Eq. (11), if $p(a) = 0$, then the boundary condition in Eq. (6a) is not required to proof the orthogonality relation, provided that y_m, y_n and their derivatives are bounded at $x = a$. Similarly, the same is valid for the boundary condition in Eq. (6b) when $p(b) = 0$. If $p(a) = p(b) = 0$, then no boundary condition at all is required. So, the orthogonality relation holds for a singular SLP such that $p(a) = 0$ or $p(b) = 0$, as the ones involving the differential equations of Legendre and Bessel, treated in subsections 2.3 and 2.4, respectively.

2.2 Trigonometric Fourier Series

The orthogonal set of functions for the trigonometric Fourier series comes from the solution of a two-point BVP involving the following linear second-order ODE

$$y'' + \lambda y = 0, \qquad (14)$$

with the following periodic boundary conditions

$$y(-L) - y(L) = 0, \qquad (15a)$$
$$y'(-L) - y'(L) = 0, \qquad (15b)$$

where $L > 0$ and λ a real parameter. For $\lambda > 0$, the Eq. (14) is the simple harmonic oscillator differential equation. The BVP consisting of Eq. (14) and the periodic boundary conditions in (15) is called *periodic Sturm-Liouville problem* (Kreyszig *et al.*, 2011) on the interval $[-L, L]$. The eigenvalues and eigenfunctions of this SLP are obtained considering three cases for the parameter λ: $\lambda = 0$, $\lambda = -\alpha^2$, and $\lambda = \alpha^2$, with $\alpha > 0$. For $\lambda < 0$, the only solution is $y = 0$, which is not an eigenfunction (Zill, 2022). Suppose that $\lambda = 0$. Here, Eq. (14) reduces to $y'' = 0$, whose solution is given by

$$y = c_1 x + c_2, \qquad (16)$$

where c_1 and c_2 are arbitrary constants. Using Eq. (15a) in Eq. (16) yields

$$-c_1 L + c_2 = c_1 L + c_2 \Leftrightarrow 2c_1 L = 0 \Leftrightarrow c_1 = 0. \qquad (17)$$

It follows from Eqs. (16) and (17) that the eigenfunctions are $y = c_2 \neq 0$. One can choose any nonzero constant but it is convenient to define $c_2 = 1/2$ such that

$$y = \frac{1}{2} \qquad (18)$$

is the desired eigenfunction. The reasons behind this choice will be explained at the end of this subsection. Finally, suppose that $\lambda = \alpha^2$. Then, Eq. (14) can be rewritten as

$$y'' + \alpha^2 y = 0$$

whose solution is given by

$$y = c_1\cos(\alpha x) + c_2\sin(\alpha x), \tag{19}$$

such that

$$y' = -c_1\alpha\sin(\alpha x) + c_2\alpha\cos(\alpha x). \tag{20}$$

Using Eq. (15a) in Eq. (19) yields

$$2c_2\sin(\alpha L) = 0. \tag{21}$$

If $c_1 = 0$ and $c_2 \neq 0$, then it follows from Eq. (21) that

$$\alpha L = n\pi \Longrightarrow \alpha = \frac{n\pi}{L}, n = 1,2,\dots \tag{22}$$

The Eq. (22) is also obtained when $c_1 \neq 0$ and $c_2 = 0$. Considering both cases in Eq. (19), one can notice that there are two linearly independent eigenfunctions corresponding to each eigenvalue,

$$\psi_n(x) = \sin\left(\frac{n\pi x}{L}\right), n = 1,2,\dots \tag{23a}$$

and

$$\phi_n(x) = \cos\left(\frac{n\pi x}{L}\right), n = 1,2,\dots \tag{23b}$$

From Eqs. (18), (23a), and (23b) one can define a set for the eigenfunctions of this SLP,

$$B = \left\{\frac{1}{2}, \cos\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \dots, \sin\left(\frac{\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \dots\right\}$$

which is an orthogonal set in $[-L, L]$ with respect to the inner product of Eqs. (1) using $w(x) = 1, a = -L$, and $b = L$. In fact, taking the inner product between ψ_n and ϕ_n, we have

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0, & m \neq n \\ L, & m = n' \end{cases} \tag{24a}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0, & m \neq n \\ L, & m = n' \end{cases} \tag{24b}$$

for $m, n \in \mathbb{N}^*$ and

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = 0$$

for all integers m and n. The orthogonality between the eigenfunction $1/2$ and the other eigenfunctions in B follows from a simple integration and, therefore, is omitted. Using the orthogonal basis B one can expand a given function f defined on the interval $[-L, L]$ as a series of eigenfunctions, as follows

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right] \tag{25}$$

whose coefficients are given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n = 0,1,2,\dots, \tag{26a}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1,2,\dots \tag{26b}$$

The $1/L$ factor in Eqs. (26a) and (26b) comes from the squared norm of the cosines and sines as indicated in Eqs. (24a) and (24b). The series in Eq. (25) is called *trigonometric Fourier series* of f on $[-L, L]$, while the coefficients obtained from Eqs. (26a) and (26b) are referred to as Euler-Fourier formulas (Zill, 2022). There is no reason to believe that the series in the right-hand side of Eq. (25) converges to f. Fourier was the first to assert, in his book *Analytical Theory of Heat* (1822), that an arbitrary function defined on the interval $(-\pi, \pi)$ could be expressed as a trigonometric series. He provided the proof for some simple functions that he needed to study the conduction of heat but doesn't develop a rigorous proof for the general case. The problem of the representation of a function by a trigonometric series remained open for more than half a century. As mentioned in subsection 1.1, only in 1829 that Dirichlet established the conditions of convergence for the Fourier series. He stated that a bounded piecewise continuous function f defined on $(-\pi, \pi)$ converges to $f(x)$ at every point x at which f is continuous and that in a point of discontinuity $x = x_0$, f converges to

$$\frac{f(x_0^+) + f(x_0^-)}{2},$$

that corresponds to the mean of the limits of f just before and after the discontinuity. Due to this fact, some texts use the symbol $\sim$ rather than $=$ to denote a correspondence between f and its series expansion (Zill, 2022).

It is possible to obtain equivalent representations of the trigonometric Fourier series that play an important role in its physical interpretation. In fact, if f is a periodic function of time *t* with period $T = 2L$ and fundamental angular frequency $\omega_0 = 2\pi/T$, then B can be interpreted as a set of sinusoids whose frequencies are integer multiples of each other, *i.e.*, they are *harmonically related* (Oppenheim *et al.*, 1997). This interpretation is very useful since it allows one to use the trigonometric Fourier series to determine the frequency components of a signal (a function of time). In this sense, until the end of this subsection, it is considered that f is a periodic signal of period $T = 2L$. In particular, there is the complex exponential Fourier series. Before we start, recall that cosines and sines can be written in terms of complex exponential functions, as follows

$$\cos(n\omega_0 t) = \frac{\exp(jn\omega_0 t) + \exp(-jn\omega_0 t)}{2}, \tag{27}$$

$$\sin(n\omega_0 t) = \frac{\exp(jn\omega_0 t) - \exp(-jn\omega_0 t)}{2j}, \tag{28}$$

where *j* denotes the imaginary unit. Substituting Eqs. (27) and (28) in Eq. (25) yields

$$\begin{aligned}
f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right] \\
&= \underbrace{\frac{a_0}{2}}_{c_0} + \sum_{n=1}^{\infty} \Bigg[\underbrace{\left(\frac{a_n - jb_n}{2}\right)}_{c_n} \exp(jn\omega_0 t) + \underbrace{\left(\frac{a_n + jb_n}{2}\right)}_{\bar{c}_n} \exp(-jn\omega_0 t)\Bigg] \\
&= c_0 + \sum_{n=1}^{\infty} \left[c_n \exp(jn\omega_0 t) + \bar{c}_n \exp(-jn\omega_0 t)\right] \\
&= c_0 + \sum_{n=1}^{\infty} \left[c_n \exp(jn\omega_0 t) + c_{-n} \exp(-jn\omega_0 t)\right] \text{since} \bar{c}_n = c_{-n} \\
&= \sum_{n=-\infty}^{\infty} c_n \exp(jn\omega_0 t),
\end{aligned} \tag{29}$$

which is called the *complex exponential Fourier series*. The formula for the coefficients c_n in Eq. (29) follows immediately from Eqs. (26a) and (26b),

$$\begin{aligned}
c_n &= \frac{a_n}{2} - j\frac{b_n}{2} \\
&= \frac{1}{T}\int_{T/2}^{-T/2} f(t)\left[\cos(n\omega_0 t) - j\sin(n\omega_0 t)\right]dt \\
&= \frac{1}{T}\int_{\frac{T}{2}}^{-\frac{T}{2}} f(t)\exp(-jn\omega_0 t)dt.
\end{aligned} \tag{30}$$

In this paper, the plots of the magnitudes and phases of c_n in Eq. (30) versus $n\omega_0$ are collectively referred to as *bilateral spectrum* since $n \in \mathbb{Z}$. Some examples of this spectrum are presented in section 3. Another very useful alternative representation of the trigonometric Fourier series is the *polar Fourier series*. To obtain this representation, it is sufficient to rewrite the linear combination of cosines and sines in Eq. (25) as follows

$$a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) = A_n \cos(n\omega_0 t + \phi_n), \qquad (31)$$

where

$$A_n = \sqrt{a_n^2 + b_n^2} \qquad (32)$$

and

$$\phi_n = -\arctan\left(\frac{b_n}{a_n}\right) + \frac{\pi}{2}(1 - \mathrm{sgn}(a_n)), \qquad (33)$$

with $n \in \mathbb{N}^*$. The symbol sgn stands for the sign function and the second term in the right hand side of Eq. (33) is a quadrant correction. Substituting Eq. (31) in Eq. (25) yields

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n), \qquad (34)$$

where $A_0 = a_0/2$. The plots of Eqs. (32) and (33) versus $n\omega_0$ are collectively referred to as *unilateral spectrum* since $n \in \mathbb{N}$ (see section 3 for practical examples). The polar Fourier series in Eq. (34) can be used to provide a 3D representation of the trigonometric Fourier series of a function f, which is one of the main graphical features of Freya, as described in Section 3.

2.3 Fourier-Legendre Series

For the case of the Fourier-Legendre series, the corresponding Sturm-Liouville problem, on the interval $[-1,1]$, is obtained by setting $p(x) = 1 - x^2$, $q(x) = 0$, $r(x) = 1$ and $\lambda = n(n+1)$, with $n \in \mathbb{N}$, in Eq. (5). Since $p(1) = p(-1) = 0$, no boundary conditions are needed, as mentioned in the end of subsection 2.1. Substituting these definitions in Eq. (5), it reduces to the Legendre's differential equation,

$$[(1 - x^2)y']' + n(n+1)y = 0. \qquad (35)$$

It is well known that the Legendre polynomials $P_n(x)$ are the solutions of Eq. (35) related to the eigenvalues $\lambda_n = n(n+1)$. In this sense, they are the eigenfunctions of this SLP and, therefore, form a orthogonal set on the interval $[-1,1]$ with respect to the inner product in Eq. (1) using $w(x) = 1, a = -1$, and $b = 1$. So,

$$\langle P_m, P_n \rangle = \int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0, & m \neq n \\ \dfrac{2}{2n+1}, & m = n \end{cases}. \qquad (36)$$

To demonstrate Eq. (36), suppose initially that $m \neq n$ and let $P_n(x)$ and $P_m(x)$ be Legendre polynomials. Then, they must satisfy Eq. (35), that is,

$$(1 - x^2)P_m''(x) - 2xP_m'(x) + m(m+1)P_m(x) = 0, \qquad (37)$$
$$(1 - x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0. \qquad (38)$$

Multiplying Eq. (37) by $P_n(x)$ and Eq. (38) by $P_m(x)$, one can obtain

$$(1 - x^2)P_n(x)P_m''(x) - 2xP_n(x)P_m'(x) + m(m+1)P_m(x)P_n(x) = 0, \qquad (39)$$
$$(1 - x^2)P_m(x)P_n''(x) - 2xP_m(x)P_n'(x) + n(n+1)P_m(x)P_n(x) = 0. \qquad (40)$$

Subtracting Eq. (40) from Eq. (39) yields

$$(1 - x^2)[P_n(x)P_m''(x) - P_m(x)P_n''(x)] - 2x[P_n(x)P_m'(x) - P_m(x)P_n'(x)] \\ + P_m(x)P_n(x)[m(m+1) - n(n+1)] = 0 \qquad (41a)$$

or

$$\frac{d}{dx}[(1 - x^2)(P_n(x)P_m'(x) - P_m(x)P_n'(x))] \\ + P_m(x)P_n(x)[m(m+1) - n(n+1)] = 0. \qquad (41b)$$

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Solving Eq. (41b) for $P_m(x)P_n(x)$ yields

$$P_m(x)P_n(x) = -\frac{1}{m(m+1)-n(n+1)}\frac{d}{dx}\left[(1-x^2)\left(P_n(x)P_m'(x)-P_m(x)P_n'(x)\right)\right]. \tag{42}$$

Integrating both sides of Eq. (42) with respect to x from -1 to 1 gives

$$\int_{-1}^{1} P_m(x)P_n(x)dx = -\left.\frac{(1-x^2)\left(P_n(x)P_m'(x)-P_m(x)P_n'(x)\right)}{m(m+1)-n(n+1)}\right|_{-1}^{1} = 0, \tag{43}$$

which concludes the first part of the demonstration. For the second part, where $m = n$, it is convenient to use the generating function of the Legendre polynomials, *i.e.*,

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n, |x| \le 1, |t| < 1. \tag{44}$$

Squaring both sides of Eq. (44) yields

$$(1-2xt+t^2)^{-1} = \sum_{n=0}^{\infty} [P_n(x)]^2 t^{2n}. \tag{45}$$

Integrating both sides of Eq. (45) with respect to x from -1 to 1 gives

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\int_{-1}^{1}[P_n(x)]^2dx\right)t^{2n} &= \int_{-1}^{1}\frac{dx}{1-2xt+t^2}\\ &= -\left.\frac{\log(1-2xt+t^2)}{2t}\right|_{-1}^{1}\\ &= -\frac{1}{2t}[\log(1-2t+t^2)-\log(1+2t+t^2)]\\ &= \frac{1}{t}[\log(1+t)-\log(1-t)]\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}+1}{n}t^{n-1}\\ &= \sum_{n=0}^{\infty}\left(\frac{2}{2n+1}\right)t^{2n}.\end{aligned} \tag{46}$$

From the comparison of the coefficients in Eq. (46), one can infer that

$$\int_{-1}^{1}[P_n(x)]^2dx = \frac{2}{2n+1}, \tag{47}$$

completing the demonstration of Eq. (36). Substituting Eq. (47) in Eq. (3) one can obtain the Fourier coefficients for the Fourier-Legendre series, given by

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx. \tag{48}$$

With the coefficients in Eq. (48) and defining the orthogonal basis as $B = \{P_n(x)\}_{n=0}^{\infty}$, the Fourier-Legendre series of a function f defined on the interval $[-1,1]$ can be written as

$$f(x) = \sum_{n=0}^{\infty} c_nP_n(x). \tag{49}$$

It is worthwhile to mention that the orthogonality of the Legendre polynomials can be extended to an arbitrary interval $[a, b]$ if one perform some shifting and scaling operations on $P_n(x)$, *i.e.*,

$$P_n^{a,b}(x) = \sqrt{\frac{2}{b-a}}P_n\left(\frac{2x-a-b}{b-a}\right), \tag{50}$$

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Since

$$\begin{aligned}\int_a^b P_n^{a,b}(x)P_m^{a,b}(x)dx &= \frac{2}{b-a}\int_a^b P_n\left(\frac{2x-a-b}{b-a}\right)P_m\left(\frac{2x-a-b}{b-a}\right)dx \\ &= \underbrace{\int_{-1}^{1} P_n(u)P_m(u)du}_{(36)}, u = \frac{2x-a-b}{b-a} \\ &= \begin{cases} 0, & m \neq n \\ \dfrac{2}{2n+1}, & m = n \end{cases}\end{aligned}$$

Eq. (50) was used to implement the Fourier-Legendre series in Freya for a bounded piecewise continuous function f defined on an interval $[a, b]$. In that case, Eqs. (48) and (49) can be rewritten as follows

$$f(x) = \sum_{n=0}^{\infty} c_n^{a,b} P_n^{a,b}(x)$$

where

$$c_n^{a,b} = \frac{2n+1}{2}\int_a^b f(x)P_n^{a,b}(x)dx.$$

2.3.1 Bonnet's recursion formula

The Legendre polynomials satisfy several recurrence formulas that are very useful for computational purposes or to proof some of their properties. In particular, there is the Bonnet's recursion formula which is stated as

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \forall n \in \mathbb{N}^*. \quad (51)$$

To demonstrate Eq. (51) it is necessary to use the generating function, rewritten here for convenience

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

Differentiating both sides of the generating function with respect to t, one can obtain

$$\underbrace{(1-2xt+t^2)^{-\frac{1}{2}}}_{(44)}(1-2xt+t^2)^{-1}(x-t) = \sum_{n=1}^{\infty} nt^{n-1}P_n(x) \quad (52a)$$

or

$$(x-t)\sum_{n=0}^{\infty} P_n(x)t^n = (1-2xt+t^2)\sum_{n=1}^{\infty} nt^{n-1}P_n(x). \quad (52b)$$

Simplifying Eq. (52b) yields

$$\sum_{n=2}^{\infty}[xP_n(x) - P_{n-1}(x)]t^n = \sum_{n=2}^{\infty}[(n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)]t^n.$$

Since the powers of t and the limits of summation in Eq. (53) are the same, one can obtain the Bonnet's recursion formula by equating their coefficients,

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x) \quad (54a)$$

or

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x). \quad (54b)$$

The Boonet's recursion formula was used in the development of Freya to calculate the Legendre polynomials $(n \geq 2)$ from $P_0(x) = 1$ and $P_1(x) = x$.

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2.4 Fourier-Bessel Series

Consider the parametric Bessel differential equation,

$$x^2y'' + xy' + (\alpha^2x^2 - n^2)y = 0, \qquad \alpha > 0, \qquad (55a)$$

with a fixed integer $n \geq 0$. This equation can be rewritten as a Sturm-Liouville equation by setting $p(x) = x, q(x) = -\frac{n^2}{x}, r(x) = x$and $\lambda = \alpha^2$ in Eq. (5), yielding

$$(xy')' + \left(\alpha^2x - \frac{n^2}{x}\right)y = 0. \qquad (55b)$$

Since Eq. (55) is a second-order homogeneous linear differential equation, the general solution is a linear combination of two linearly independent solutions,

$$y = c_1J_n(\alpha x) + c_2Y_n(\alpha x), \qquad n = 0,1,2, \ldots \qquad (56)$$

where $J_n(\alpha x)$ and $Y_n(\alpha x)$ are scaled versions of the Bessel functions of the first and second kinds, and c_1 and c_2 are constants. Since the functions $Y_n(\alpha x)$ are not bounded at $x = 0$, we will consider only the functions $J_n(\alpha x)$ in such a way that Eq. (56) is rewritten as

$$y = J_n(\alpha x), \qquad n = 0,1,2, \ldots \qquad (57)$$

Since $p(0) = 0$, the boundary condition in equation (6a) is not required. On the other hand, given a fixed real number b one can form a Sturm-Liouville problem on $[0, b]$ using Eq. (55b) together with the following boundary condition at $x = b$,

$$\beta_1y(b) + \beta_2y'(b) = 0.$$

There are three cases for this boundary condition but in this discussion only the one in which $\beta_1 = 1$ and $\beta_2 = 0$ is considered. This will simplify the formula for the coefficients in the Fourier-Bessel series. So, the solutions in Eq. (57) must satisfy

$$J_n(\alpha b) = 0. \qquad (58)$$

It can be shown that Eq. (58) has an infinite number of positive roots x_i such that

$$\alpha b = x_i \Longrightarrow \alpha_i = \frac{x_i}{b}, \qquad i = 1,2, \ldots$$

In this sense, the eigenvalues are positive and given by $\lambda_i = \alpha_i^2 = x_i^2/b^2$ and the corresponding eigenfunctions are $J_n(\alpha_ix)$. Thus, the set $B = \{J_n(\alpha_ix)\}_{i=1}^{\infty}$ is orthogonal with respect to the inner product of Eq. (1) using $w(x) = r(x) = x$ on an interval $[0, b]$, *i.e.*,

$$\int_0^b xJ_n(\alpha_ix)J_n(\alpha_jx)dx = \begin{cases} 0, & i \neq j \\ \frac{b^2}{2}J_{n+1}^2(\alpha_ib), & i = j \end{cases}. \qquad (59)$$

To obtain Eq. (59) set $u = J_n(\alpha_ix)$ and $v = J_n(\alpha_jx)$. So, it follows from Eq. (55a) that

$$x^2u'' + xu' + (\alpha_i^2x^2 - n^2)u = 0, \qquad (60)$$
$$x^2v'' + xv' + (\alpha_j^2x^2 - n^2)v = 0. \qquad (61)$$

Multiplying Eq. (60) by v/x and (61) by u/x and subtracting the latter from the former, yields

$$(\alpha_j^2 - \alpha_i^2)xuv = -\frac{d}{dx}[x(vu' - uv')]. \qquad (62)$$

Integrating both sides of Eq. (62) with respect to x from 0 to b, gives

$$\int_0^b xJ_n(\alpha_ix)J_n(\alpha_jx)dx = -b\frac{J_n(\alpha_ib)\alpha_jJ_n'(\alpha_jb) - J_n(\alpha_jb)\alpha_iJ_n'(\alpha_ib)}{\alpha_j^2 - \alpha_i^2}. \qquad (63)$$

The Eq. (63) vanishes when $i \neq j$ since $J_n(\alpha_ib) = J_n(\alpha_jb) = 0$. For $i = j$, it reduces itself to an indeterminate form. So, to overcome this problem one should consider a limit where $\alpha_i \to \alpha_j$ and then apply the L'Hospital's rule, as follows

$$\begin{aligned}
\int_0^b xJ_n(\alpha_i x)J_n(\alpha_j x)dx &= -b \lim_{\alpha_j\to\alpha_i} \frac{J_n(\alpha_i b)\alpha_j J_n'(\alpha_j b) - J_n(\alpha_j b)\alpha_i J_n'(\alpha_i b)}{\alpha_j^2 - \alpha_i^2} \\
&= -b \lim_{\alpha_j\to\alpha_i} \frac{-\alpha_i J_n(\alpha_j b)J_n'(\alpha_i b)}{\alpha_j^2 - \alpha_i^2} \\
&= -b^2 \lim_{\alpha_j\to\alpha_i} \frac{-\alpha_i J_n'(\alpha_j b)J_n'(\alpha_i b)}{2\alpha_j} \\
&= \frac{b^2}{2}[J_n'(\alpha_i b)]^2 \\
&= \frac{b^2}{2}[J_{n+1}(\alpha_i b)]^2,
\end{aligned}$$

completing the proof of Eq. (59). Therefore, the *Fourier-Bessel series* of a function f defined on the interval $[0, b]$ is given by

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x),$$

where

$$c_i = \frac{2}{b^2 J_{n+1}^2(\alpha_i b)} \int_0^b xJ_n(\alpha_i x)f(x)dx$$

are the series coefficients.

3. Results and Discussion - Freya

Freya is an educational software package developed in MATLAB® with a graphical user interface (GUI) to provide a friendly and visual approach for the subject of generalized Fourier series in lectures on signal processing, linear algebra, and ordinary differential equations, to name just a few. It is designed to help students to develop intuition and to understand the underlying idea of the subject before delving into mathematics, which can be quite useful for those who are seeing the subject for the first time.

3.1 Overview of Freya Interface

Figure 2 shows the main window of Freya, which is divided into two panels: the left one (*Function Configuration*) allows the users to set up the functions that will be expanded in a generalized Fourier series, while the right one (*Fourier Series Visualization*) is for the visualization of both the provided function and its Fourier series along with the coefficients.

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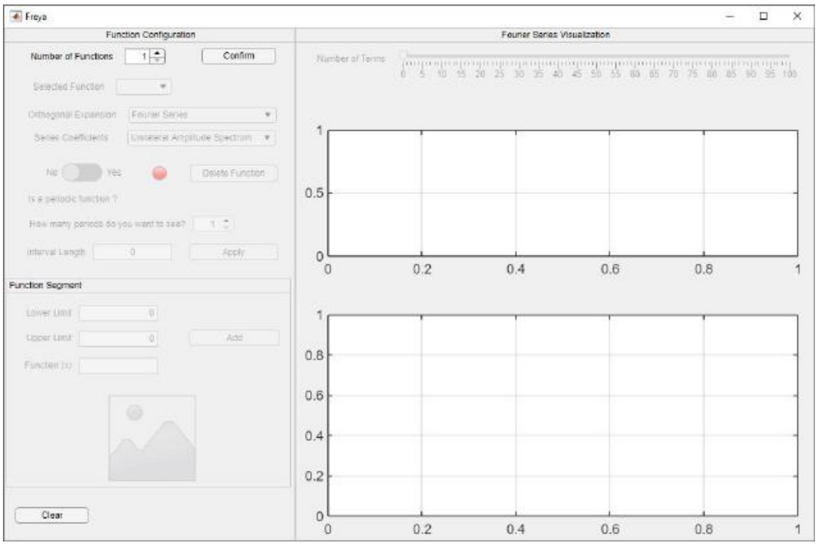
Figure 2 - The startup screen of Freya, a MATLAB GUI-based tool designed to be a learning aid system for students as well as for teaching.

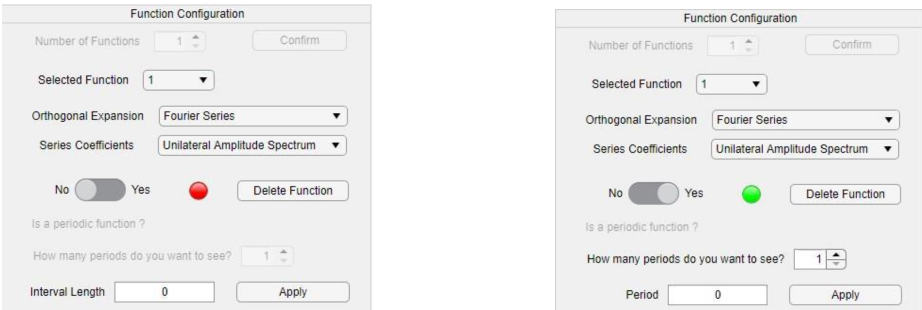


Source: Authors.

3.1.1 Function Configuration

The widgets in *Function Configuration* panel are used to specify the functions to be expanded in any of the three specific cases of the generalized Fourier series discussed in Section 2. The user has to inform the number of functions in the "Number of Functions" spinner, which goes from 1 up to 10. By pressing the "Confirm" button, some widgets are enabled such as the "Selected Function", "Orthogonal Expansion", and "Series Coefficients" drop-down menus, as illustrated in Figure 3a.

Figure 3 - The *Function Configuration* panel has several features that allow the user to set up a function and the corresponding series expansion.



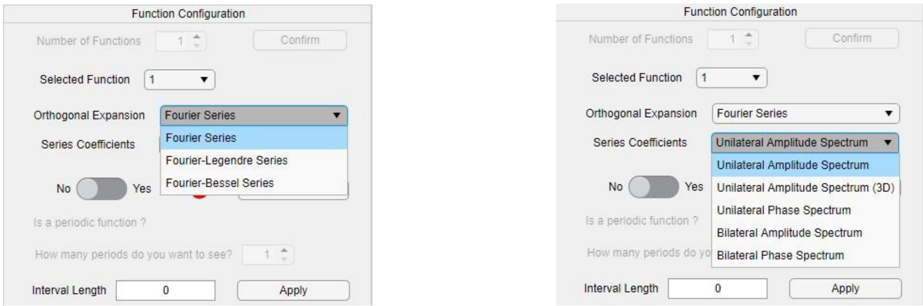
(a) When the user press the "Confirm" button, some widgets of the panel are enabled.

(b) The user can specify if the function is periodic by using the "Is a periodic function ?" switch.

Source: Authors.

The first one is filled with integers from 1 up to the number of functions, allowing the user to set up a specific function. The second one contains a list of available Fourier series expansions, which in Freya's current version are the trigonometric Fourier series, Fourier-Legendre series, and Fourier-Bessel series, as indicated in Figure 4a. The third one is a list of the coefficients that is updated based on the chosen series since each one has its Fourier coefficients; Figure 4b shows the coefficients for the trigonometric case.

Figure 4 - These drop-down menus allow the user to choose the Fourier series for which the function will be expanded and the associated coefficients to be shown in the lower plot of the visualization panel.



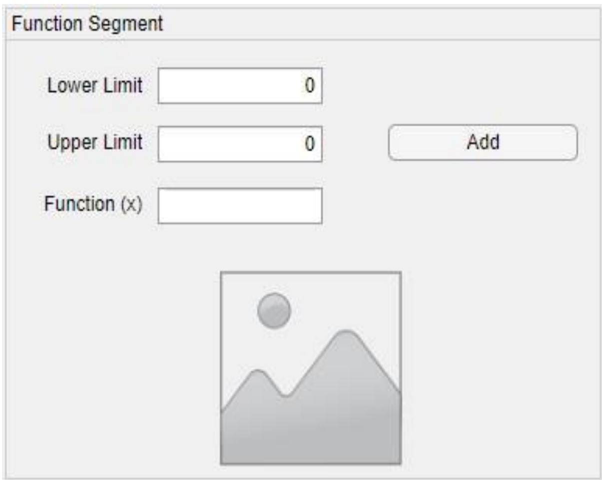
(a) Drop-down menu showing the available Fourier series expansions.

(b) Drop-down menu with the coefficients for the trigonometric case.

Source: Authors.

An interesting feature is that Freya can handle periodic functions. The users can configure this property in the "Is a periodic function?" switch, which is also enabled with the drop-down menus just described. As depicted in Figures 3a and 3b, by choosing "Yes" the red lamp becomes green and the "Number of Periods" spinner is enabled to determine the number of periods to be shown in the upper plot of the visualization panel. After this, the user should fill the "Interval Length" (or "Period" when the function is periodic) field and press the "Apply" button to enable the "Function Segment" sub-panel, as shown in Figure 5.

Figure 5 - The *Function Segment* sub-panel allows the user to set up each segment by specifying the function rule and its corresponding interval.



Source: Authors.

As noticed in Figure 5, this sub-panel allows the user to set up a function segment by entering the function rule (in the *x* variable) through the "Function (*x*)" field and the corresponding interval in the "Lower Limit" and "Upper Limit" fields. The provided function segment is applied by pressing the "Add" button. Each time this button is pressed, an image is shown at the bottom to remind the user of all function segments provided so far. If for any reason the user provides a lower limit that is greater than the upper limit, a warning message arises, as can be seen in Figure 6a. An error message appears when the user provides a function rule that is not in the *x* variable, as indicated in Figure 6b.

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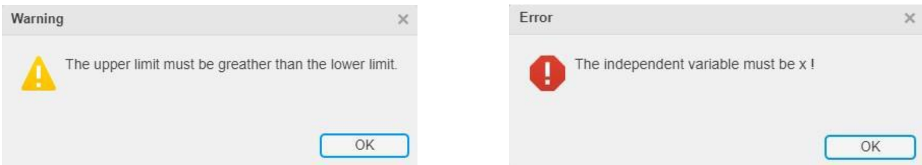
Figure 6 - Some dialog boxes that an user can experience using Freya.



(a) Warning dialog box that arises when the user provides a lower limit greater than the upper limit of a function segment

(b) Warning dialog box that arises when the user provides a function rule that is not in the *x* variable

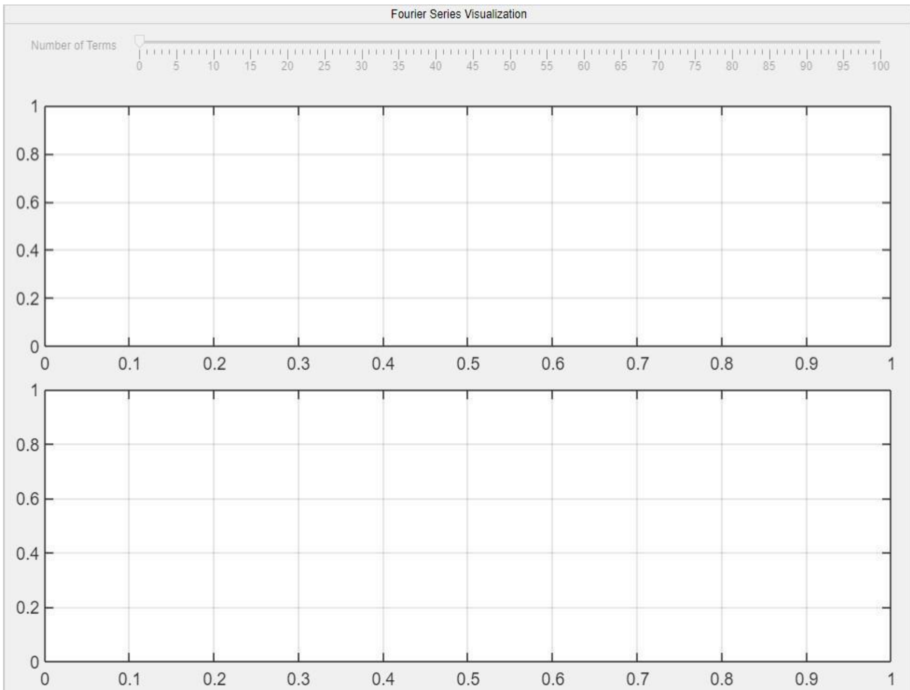
Source: Authors.

The user can delete a specific function using the "Delete Function" button in Figure (3a) or can delete all functions using the "Clear" button in the "Function Configuration" panel, at the bottom of Figure 2.

3.1.2 Fourier Series Visualization

In the upper plot of the *Fourier Series Visualization* the user can see the provided function and its corresponding Fourier series expansion overlaid on the same plot. The number of terms of the partial sums can be changed using the "Number of Terms" slider bar, which goes from 1 up to 100, as depicted in Figure 7. The lower plot shows the coefficients indicated in the "Series Coefficients" drop-down menu. For the trigonometric case, the coefficients compose a spectrum which covers both amplitude and phase. The following subsection discuss the Freya functionality for a simple example that will clarify how this panel works.

Figure 7 - In the *Fourier Series Visualization* panel the user can see the plot of both the provided function and the chosen Fourier series expansion in the upper plot, while the lower plot contains the coefficients chosen in the "Series Coefficients" drop-down menu. The number of terms of the partial sums of the expansion can be changed using the "Number of Terms" slider bar.



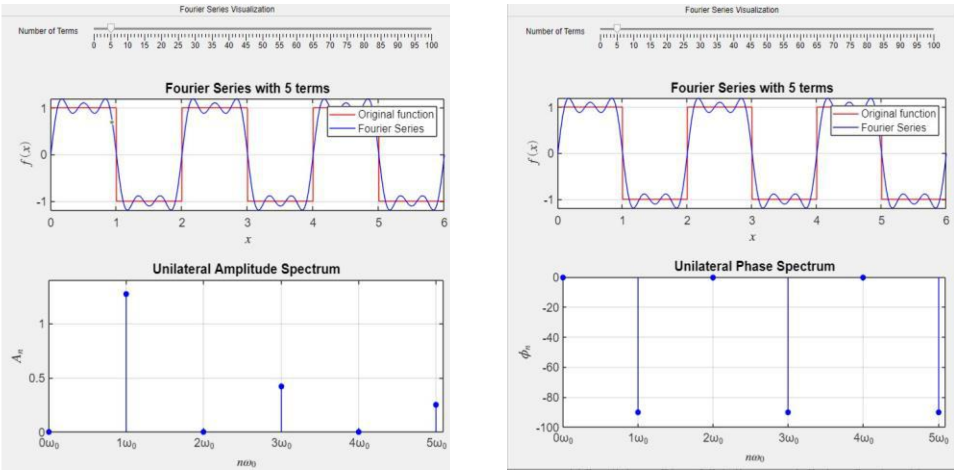
Source: Authors.

3.2 Using the Tool

In this subsection, the use of Freya is presented through its application to common functions such as square, triangle, and sawtooth waves. Each of these functions is expanded using the three specific cases of the generalized Fourier series described in Section 2. For the trigonometric case, a description of the functions in terms of their frequency components is given by using both unilateral and bilateral spectrum.

Figure 8 shows Freya for a periodic square wave of period $T = 2$ which has the value 1 on the interval $(0,1)$ and the value -1 on the interval $(1,2)$. At the *Fourier Series Visualization* panel there are two plots, the upper one is the trigonometric Fourier series representation (in blue) of the periodic square wave (in red) considering only the first five terms of the series, as set up by the user in the "Number of Terms" slider bar. The lower one is the unilateral amplitude spectrum since this option was selected in the "Series Coefficients" drop-down menu. The number of harmonics shown by the spectrum is the same as the number of terms. Note that the lower frequencies are the most important in the composition of the provided function since as n goes to infinity the amplitude tends to zero. Also, note that only three periods are shown in the upper plot but the user can change this using the "Number of Periods" spinner.

Figure 8 - Freya's "Fourier Series Visualization" panel showing the trigonometric Fourier series of a periodic square wave of period $T = 2$ for the first 5 terms.



(a) Unilateral Amplitude Spectrum **(b)** Unilateral Phase Spectrum

Source: Authors.

To obtain the unilateral spectrum depicted in Figure 8 one must use equations (32) and (33), which in turn requires the trigonometric Fourier series coefficients of the periodic square wave. Sometimes the computation of the coefficients can be cumbersome but symmetry can be used as an aid to simplify the integration process. For example, since the considered periodic square wave is an odd function, it follows that $a_n = 0$ for $n \geq 0$. In this sense, only the b_n coefficients need to be determined by using equation (26b). So, the coefficients of the periodic square wave are given by

$$a_n = 0, \qquad n = 0,1,2,\ldots, \tag{64}$$

$$b_n = \frac{2}{n\pi}[1 - \cos(n\pi)], \qquad n = 1,2,\ldots. \tag{65}$$

Substituting equations (64) and (65) in equations (32) and (33) one can obtain

$$A_n = \begin{cases} \dfrac{4}{n\pi}, & n = 1,3,5,\ldots \\ 0, & n = 0,2,4,6,\ldots \end{cases} \tag{66}$$

and

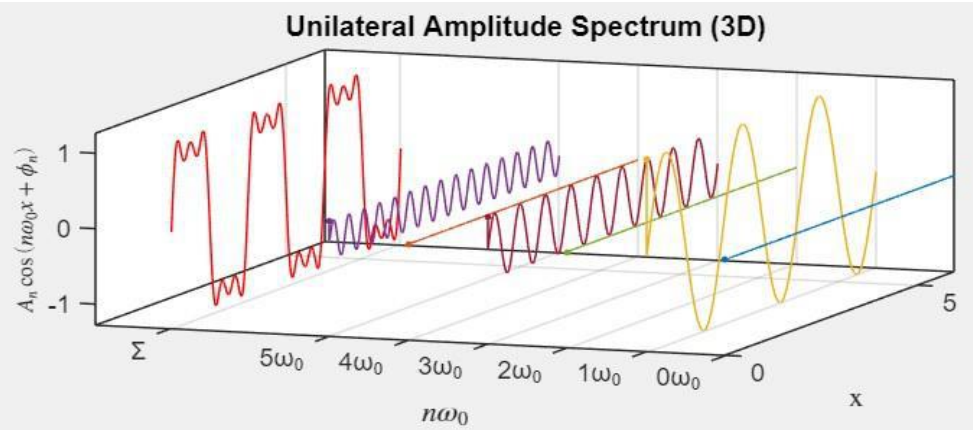
$$\phi_n = \begin{cases} -\frac{\pi}{2}, & n = 1,3,5, \ldots \\ 0, & n = 0,2,4,6, \ldots \end{cases}. \tag{67}$$

As one can notice, equation (66) is in accordance with the unilateral amplitude spectrum shown in Figure 8a since the even harmonics are zero and the odd ones decay as $1/n$. Similarly, equation (67) describes the unilateral phase spectrum in Figure 8b. Using equations (66) and (67) in the polar Fourier series (equation (34)) yields

$$\begin{aligned} f(t) &= \sum_{n=1, n \text{ odd}}^{\infty} \frac{4}{n\pi} \sin(n\pi t) \\ &= \frac{4}{\pi} \sin \pi t + \frac{4}{3\pi} \sin 3\pi t + \frac{4}{5\pi} \sin 5\pi t + \frac{4}{7\pi} \sin 7\pi t + \cdots \end{aligned} \tag{68}$$

As suggested by equation (68), the periodic square wave is written in terms of individual sine waves, where each one has a well-defined amplitude and frequency. The most important frequencies are associated to the sine waves with the highest amplitudes. Figure 8 shows that the lower frequencies are the most relevant for the periodic square wave. In fact, this is the power of the trigonometric Fourier series and its equivalent versions since one can analyze a signal in the frequency domain to determine the frequencies that compose it. To explore this interpretation, Freya has a graphical feature that allows the user to see both the time and the frequency domains, as shown in Figure 9. To generate this spectrum, the user must select the option “Unilateral Amplitude Spectrum (3D)” in the “Series Coefficients” drop-down menu.

Figure 9 - The decomposition of the periodic square wave in sine waves, each one with a well-defined amplitude and frequency that allows the user to analyze the signal in the frequency domain.



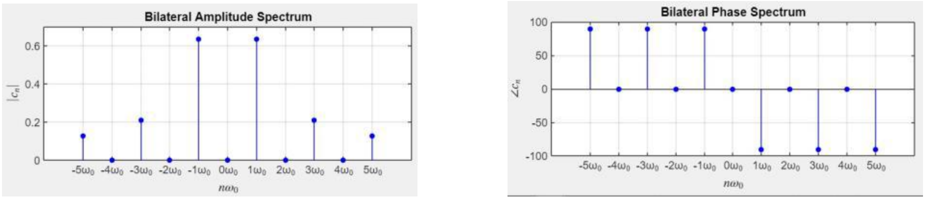
Source: Authors.

As for the unilateral case, one can obtain the bilateral frequency spectrum using equations (64) and (65) in equation (29), yielding

$$c_n = -j\frac{2}{n\pi}, n = \pm 1, \pm 3, \pm 5, \ldots. \tag{69}$$

Note that the complex coefficients in equation (69) are odd since $c_{-n} = -c_n$ holds for every integer. This means that the bilateral amplitude and phase spectra are even and odd, respectively. The user can be sure of this by selecting these spectra in the “Series Coefficients” drop-down menu. Figures 10a and 10b illustrate both spectra generated by Freya.

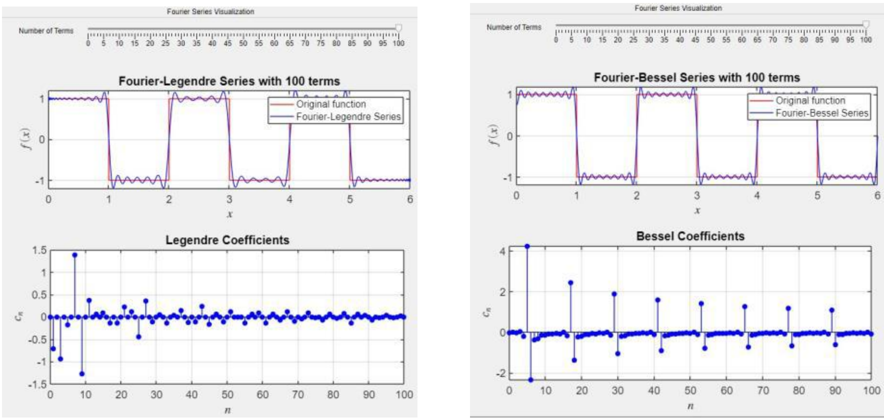
Figure 10 - The bilateral frequency spectrum of the periodic square wave. Note that the amplitude spectrum is even, while the phase spectrum is odd.



(a) Bilateral Amplitude Spectrum **(b)** Bilateral Phase Spectrum
Source: Authors.

As mentioned in Subsection 3.1, Freya's current version allows the user to expand a function using the Fourier-Legendre and Fourier-Bessel series, that can be chosen in the "Orthogonal Expansion" drop-down menu, as illustrated in Figure 4a. Figure 11 shows both expansions for the periodic square wave. Note that the Fourier-Bessel series converges faster to the original function than Fourier-Legendre series.

Figure 11 - Fourier-Legendre and Fourier-Bessel series of the periodic square wave. In each case, the upper plot shows the original function and its corresponding expansion considering 100 terms, while the lower plot shows the series coefficients.



(a) Fourier-Legendre series **(b)** Fourier-Bessel series
Source: Authors.

4. Conclusion

This paper describes Freya, a graphical user interface (GUI) built-in MATLAB® App Designer environment to be an educational tool for generalized Fourier series. The relationship between the generalized Fourier series and the Sturm-Liouville theory is briefly discussed to present the reader a method to obtain specific basis functions to be used in an orthogonal expansion. More precisely, three specific Sturm-Liouville problems are considered, yielding in the trigonometric Fourier series, Fourier-Legendre series and Fourier-Bessel series, that compose the underlying mathematical theory needed for the development of Freya. In the future, the authors intend to improve Freya by adding other orthogonal expansions such as those based on Chebyshev, Hermite, and Laguerre polynomials as well as some mechanism to compare these expansions.

Code availability

Freya is made available to the users at https://github.com/HumbertoGimenesMacedo/Freya.

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